THE APPLICATION OF NONLINEAR PROGRAMMING METHODS TO THE SOLUTION OF CONSTRAINED SADDLE-POINT PROBLEMS

by

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THESIS

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ABSTRACT

Nonlinear programming methods are used to solve saddle-point problems subject to inequality constraints on the variables; in particular, the type of saddle-point problem arising in pursuit-evasion differential games is considered. The methods investigated fall into two groups: solution of the nonlinear simultaneous equations obtained from the Kuhn-Tucker conditions, and solution of a sequence of constrained optimization problems by the gradient projection algorithm. These methods are applicable to any real-valued function $f(\underline{x},\underline{y})$ which is convex in $\underline{x}$, concave in $\underline{y}$, and has continuous and bounded second partial derivatives. Several examples are given which illustrate the characteristics of the numerical procedures.

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$$J = \int_0^1 [(x + 2t)^2 + xyt - (y + 3t)^2]\,dt$$

subject to $\int_0^1 x\,dt = 1$, $x \geq 0$, $\int_0^1 y\,dt = 1$, $y \geq 0$.

I. INTRODUCTION

The original development of differential game theory is credited to Isaacs for his work during the 1950's [8]. His book [9], published in 1965, provides a comprehensive summary of this theory. Issacs' development closely resembles the dynamic programming approach to optimal control theory popularized by Bellman [2]. Many subsequent authors have studied differential games; among the significant contributors are Ho, Bryson, and Baron [6], and Berkowitz [3]. These and many recent papers, use the calculus of variations as an analytical tool.

A statement of the differential game problem is as follows: A dynamic system is described by the state equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}(t),\ \underline{u}(t),\ \underline{v}(t),\ t) \quad ; \quad \underline{x}(t_o) = \underline{x}_o \tag{1}$$

where $\underline{x}$ is a vector which describes the position, or state, of the game, $\underline{u}$ is a control vector selected by player 1 and $\underline{v}$ is a control vector selected by player 2. The vectors $\underline{u}$ and $\underline{v}$ are members of the sets U and V respectively, denoted by $\underline{u} \epsilon U$ and $\underline{v} \epsilon V$, where U and V are to be specified later. The performance criterion is a functional assumed to be of the form

$$J = h(\underline{x}(t_f),\ t_f) + \int_{t_o}^{t_f} g(\underline{x}(t),\ \underline{u}(t),\ \underline{v}(t),\ t)dt, \tag{2}$$

which player 1 wishes to maximize, and player 2 wishes to minimize. A saddle point of J is sought (if it exists), i.e., a $\underline{u}^* \epsilon U$ and $\underline{v}^* \epsilon V$ are sought such that

$$J(\underline{u},\underline{v}^*) \leq J(\underline{u}^*, \underline{v}^*) \leq J(\underline{u}^*, \underline{v}) \quad (3)$$

In the terminology of game theory, J is the payoff, $\underline{u}$ and $\underline{v}$ are strategies of the two opposing players, and $\underline{u}^*$ and $\underline{v}^*$ are optimal pure strategies. A strategy is a decision rule that specifies a control to be applied by a player, based on all information available to him, in any given situation. Generally, the literature has considered a strategy to be a feedback control law, that is, $\underline{u}$ and $\underline{v}$ are specified in terms of the states $\underline{x}$; however, Willman [20] has used the concept of an open-loop strategy. In terms of the previous definition, this means that the information available to each player is restricted to the initial state and time, and the terminal time. Note that $\underline{u}^*$ and $\underline{v}^*$ are minimax strategies in the sense that player 1 maximizes his minimum gain while player 2 minimizes his maximum loss.

The concept of fixed terminal time ignores the fact that in more general games the game may not terminate at all. Berkowitz [4] requires a differential game to terminate whenever t and $\underline{x}(t)$ are such that $(t,\underline{x}(t))$ is a point of a previously specified set in $(t,\underline{x})$ space. This is the "terminal surface" discussed by Isaacs [9]. Berkowitz [4] also restricts the term "differential game" to include only two-person, zero-sum games (the type described above), although others [19] have not been so restrictive and have extended the theory to N-player, non-zero-sum games. Zero-sum refers to the fact that there is a single performance criterion which one player tries to maximize while the other tries to minimize. Thus, the maximizing player's gain is the minimizing player's loss and the sum of this gain and loss is zero. In a non-zero-sum game, the objectives of the two players are not directly opposed -- each may have an individual performance measure to minimize, and the sum of the two players' criteria is not necessarily zero. It may also be that there are more than two players, each controlling an input to a single system, and each trying to minimize his individual performance criterion. Such problems are not considered in this thesis.

Although variational techniques seem to be an appropriate method for solving differential games, a distinct drawback is that inequality constraints on the value of the state and control variables at any time are difficult to include in the analysis [4]. Thus, more tractable methods are sought. Since the problem is basically an optimization problem subject to inequality constraints it is natural to consider mathematical programming techniques (nonlinear programming in particular since the performance measure will generally be nonlinear). As far as this author can determine, only one other paper [10] has considered this approach.

The nonlinear programming problem was first stated mathematically by Kuhn and Tucker [14]. The problem is to find an $\underline{x}^{\circ}$ that maximizes a function $F(\underline{x})$ subject to inequality constraints of the form $\underline{\lambda}(\underline{x}) \geq \underline{0}$[1] and $\underline{x} \geq \underline{0}$. Kuhn and Tucker found necessary conditions for $\underline{x}^{\circ}$ to be a solution to this problem. If $F(\underline{x})$ is a concave function in the region where $\underline{\lambda}(\underline{x}) \geq \underline{0}$ and $\underline{x} \geq \underline{0}$, the conditions are also sufficient. Several algorithms to solve this problem have been proposed although this thesis will use only Rosen's gradient projection algorithm [15].

---

[1] $\underline{\lambda}(\underline{x}) \geq \underline{0}$ means that each component of the vector $\underline{\lambda}(\underline{x})$ is nonnegative.

The use of nonlinear programming methods to solve optimal control problems has been discussed by Rosen [16], [17], and by Kirk [12]. This thesis will discuss the connection between nonlinear programming and saddle-point problems such as arise in pursuit-evasion differential games, and several examples which illustrate these ideas will be solved.

II. BASIC PRINCIPLES

Several results from Game Theory will be required and are stated here. A more complete discussion of these theorems and their proofs can be found in [11] .

Theorem 1. General Min-Max Theorem. Let $f(\underline{x}, \underline{y})$ be a real-valued function of two vectors $\underline{x}$ and $\underline{y}$ which lie in X and Y, respectively, where both X and Y are closed, bounded, convex sets.[2] If f is continuous, convex in $\underline{x}$ for each $\underline{y}$, and concave in $\underline{y}$ for each $\underline{x}$,[3] then

$$\min_{\underline{x}\in X} \max_{\underline{y}\in Y} f(\underline{x},\underline{y}) = \max_{\underline{y}\in Y} \min_{\underline{x}\in X} f(\underline{x},\underline{y}) \quad .$$

A sufficient condition for a saddle point follows from the next theorem.

Theorem 2. A necessary and sufficient condition for

$$\min_{\underline{x}\in X} \max_{\underline{y}\in Y} f(\underline{x},\underline{y}) = \max_{\underline{y}\in Y} \min_{\underline{x}\in X} f(\underline{x},\underline{y}) \quad .$$

is that there exist a pair $\underline{x}^*\in X$, $\underline{y}^*\in Y$ such that for all $\underline{y}$ in Y and all $\underline{x}$ in X

---

[2] X is a convex set if the straight line joining any two points in X lies entirely within X. That is, if $\underline{x}^1$, $\underline{x}^2 \in X$ then

$$\underline{x}^3 = \theta\underline{x}^1 + (1-\theta)\underline{x}^2 \in X \text{ for all } 0\le\theta\le 1 \quad .$$

[3] $f(\underline{x})$ is a convex function if

$$(1-\theta)f(\underline{x}^0) + \theta f(\underline{x}^1) \ge f([1-\theta]\underline{x}^0 + \theta\underline{x}^1) \qquad \text{(i)}$$

for $0 \le \theta \le 1$ and for all $\underline{x}^0$ and $\underline{x}^1$. The negative of a convex function is a concave function, i.e., $f(\underline{x})$ is a concave function if the sign of the inequality in (i) is reversed.

$$f(\underline{x}^*, \underline{y}) \leq f(\underline{x}^*, \underline{y}^*) \leq f(\underline{x}, \underline{y}^*) \quad . \tag{4}$$

Thus, a sufficient condition for the existence of a saddle point (in the sense of (4) above) is that $f(\underline{x}, \underline{y})$ be convex in $\underline{x}$ for all $\underline{y}$ and concave in $\underline{y}$ for all $\underline{x}$.

Considering each side of the double inequality in (4) separately, the problem can be viewed as two optimization problems. Looking at the right-hand inequality first, and specifying the set Y as $Y = \{\underline{y} \mid \lambda_{yi}(y) \geq 0, i=1, \ldots, \ell\}$, the problem

$$\max_{\underline{y} \in Y} f(\underline{x}^*, \underline{y})$$

where $\underline{y}$ is an m-vector, is a statement of a Maximum Problem from non-linear programming. If $f(\underline{x}, \underline{y})$ has continuous first partial derivatives, the Kuhn-Tucker necessary conditions can be written.[4] For $\underline{y}^*$ to be a solution to the above problem it is necessary that $\underline{y}^*$ and the vector of Lagrange multipliers $\underline{\mu}_y$ satisfy the following conditions

$$\nabla_{\underline{y}} f(\underline{x}^*, \underline{y}^*) + \sum_{i=1}^{\ell} [\mu_{yi} \nabla_{\underline{y}} \lambda_{yi} (\underline{y}^*)] = \underline{0} \tag{5}$$

$$\underline{\lambda}_y (\underline{y}^*)^T \underline{\mu}_y = 0 \tag{6}$$

$$\underline{\lambda}_y (\underline{y}^*) \geq \underline{0} \tag{7}$$

$$\underline{\mu}_y \geq \underline{0} \tag{8}$$

[4] The original problem considered by Kuhn and Tucker also included a nonnegativity constraint on the variables, $\underline{y} \geq \underline{0}$, but general usage has associated the term Kuhn-Tucker conditions with the necessary conditions for this problem as well as the one they originally considered.

where $\nabla_{\underline{y}} f(\underline{x}^*, \underline{y}^*)$ denotes the gradient of $f(\underline{x}^*, \underline{y})$ with respect to $\underline{y}$ evaluated at $\underline{x}^*$, $\underline{y}^*$, i.e.,

$$\nabla_{\underline{y}} f(\underline{x}^*, \underline{y}^*) = \left.\begin{bmatrix} \dfrac{\partial f(\underline{x}^*, \underline{y})}{\partial y_1} \\ \vdots \\ \dfrac{\partial f(\underline{x}^*, \underline{y})}{\partial y_m} \end{bmatrix}\right|_{\underline{y} = \underline{y}^*} ,$$

and that the constraint qualification [14] be satisfied. The constraint qualification rules out pathological behavior on the boundary of the constraint set. In this thesis, only linear constraints will be considered, hence the constraint qualification is always satisfied. It has been shown [14] that if $f(\underline{x}^*, \underline{y})$ is a concave function of $\underline{y}$, then the conditions (5)-(8) are also sufficient.

Similarly, for the problem,

$$\min_{\underline{x} \epsilon X} f(\underline{x}, \underline{y}^*)$$

$$\text{subject to } \lambda_{xi}(\underline{x}) \geq 0 \quad , \quad i=1,2,\ldots,k$$

with f a convex differentiable function of $\underline{x}$, an n-vector, the necessary and sufficient conditions are,

$$-\nabla_{\underline{x}} f(\underline{x}^*, \underline{y}^*) + \sum_{i=1}^{k} [\mu_{xi} \nabla_{\underline{x}} \lambda_{xi}(\underline{x}^*)] = \underline{0} \tag{9}$$

$$\underline{\lambda}_x (\underline{x}^*)^T \underline{\mu}_x = 0 \tag{10}$$

$$\underline{\lambda}_x(\underline{x}^*) \geq \underline{0} \tag{11}$$

$$\underline{\mu}_x \geq \underline{0} \tag{12}$$

where $\underline{\mu}_x$ is a vector of Lagrange multipliers. Thus, for $(\underline{x}^*, \underline{y}^*)$ to be a solution of the constrained saddle-point problem, equations (5), (6), (9), and (10) and inequalities (7), (8), (11), and (12) must be satisfied. Although the Kuhn-Tucker conditions require only that the function be real-valued, concave-convex, and differentiable to ensure that a saddle point exists, the methods examined in this thesis will also require the function to have continuous and bounded second partial derivatives.

III. NUMERICAL METHODS

The methods investigated for solving the constrained saddle-point problem may be divided into two groups; 1) solution of the nonlinear simultaneous equations given by the Kuhn-Tucker conditions, and 2) direct solution of a maximization problem and a minimization problem. Methods in the first group have the disadvantage that although the original problem is of dimension m+n, the Langrange multipliers must also be treated as variables and the dimensionality increases to m+n+k+l.

A. SOLUTION OF NONLINEAR SIMULTANEOUS EQUATIONS

1. Newton-Raphson Method

Jacob and Polak [10] suggest a generalized Newton-Raphson method for solving the system of four(vector) equations (5), (6), (9), and (10) subject to the four(vector) inequalities (7), (8), (11), and (12). Convergence of this method was found to be somewhat sensitive to the starting point, although convergence was quite rapid if appropriate starting points were chosen.

2. Brown's Algorithm[5]

Brown [5] has proposed an algorithm to solve nonlinear simultaneous equations which requires fewer multiplications than the Newton-Raphson method. It is quite similar to the Gauss-Seidel process for nonlinear systems of equations. Brown has observed that the stability

---

[5] Subroutine NLNSYS at NPS Computer Facility.

and convergence of the algorithm do not seem to be dependent on the exact evaluation of the partials and first differences may be used to approximate the partials. This method was also found to be sensitive to the starting point.

3. Gradient Projection Determination of the Roots of the Equations given by the Necessary Conditions[6]

This method suggested itself due to the nonnegativity constraints on the multipliers in the necessary conditions. There are several ways of formulating this problem; the one chosen was to minimize

$$J = [\underline{\lambda}_x(\underline{x})^T \underline{\mu}_x]^2 + [\underline{\lambda}_y(\underline{y})^T \underline{\mu}_y]^2 \tag{13}$$

subject to

$$\begin{aligned} -\nabla_{\underline{x}} f(\underline{x},\underline{y}) + \sum_{i=1}^{k} \mu_{xi} \nabla_{\underline{x}} \lambda_{xi}(\underline{x}) &= \underline{0} \\ \nabla_{\underline{y}} f(\underline{x},\underline{y}) + \sum_{i=1}^{\ell} \mu_{yi} \nabla_{\underline{y}} \lambda_{yi}(\underline{y}) &= \underline{0} \\ \underline{\lambda}_x(\underline{x}) &\geq \underline{0} \\ \underline{\lambda}_y(\underline{y}) &\geq \underline{0} \\ \underline{\mu}_x &\geq \underline{0} \\ \underline{\mu}_y &\geq \underline{0} . \end{aligned} \tag{14}$$

Although the gradient projection algorithm can be altered to include equality constraints [15], they may also be written as two inequalities, i.e.

---

[6] The gradient projection algorithm programmed by D. E. Kirk was used in this and the following two applications.

$$\underline{0}_- \le [-\nabla_{\underline{x}} f(\underline{x},\underline{y}) + \Sigma_i \mu_{xi} \nabla_{\underline{x}} \lambda_{xi}(\underline{x})] \le \underline{0}_+$$
$$\underline{0}_- \le [\nabla_{\underline{y}} f(\underline{x},\underline{y}) + \Sigma_i \mu_{yi} \nabla_{\underline{y}} \lambda_{yi}(\underline{y})] \le \underline{0}_+ \tag{15}$$

The minimum value of f will be zero and the values of $\underline{x}$, $\underline{y}$, $\underline{\mu}_x$, $\underline{\mu}_y$ yielding this minimum will satisfy the Kuhn-Tucker necessary conditions.

B. DIRECT MINIMIZATION AND MAXIMIZATION BY GRADIENT PROJECTION

1. Gradient Projection Solution of a Sequence of Constrained Optimization Problems

Since the necessary conditions in Rosen's algorithm are equivalent to the Kuhn-Tucker conditions (see Appendix A), this method may be described in terms of two optimization problems. An arbitrary $\underline{y}$, say $\underline{y}_o$, was chosen and the minimization problem

$$\min_{\underline{x} \epsilon X} f(\underline{x},\underline{y}_o) \tag{16}$$

subject to

$$\underline{\lambda}_x(\underline{x}) \ge \underline{0} \quad ,$$

was solved for $\underline{x}_1$. Then a solution $\underline{y}_1$ was found for the maximization problem

$$\max_{\underline{y} \epsilon Y} f(\underline{x}_1, \underline{y}) \tag{17}$$

subject to

$$\underline{\lambda}_y(\underline{y}) \ge \underline{0} \quad .$$

This procedure was continued until some stopping criterion was met. A convenient stopping criterion is

$$[f(\underline{x}_i,\underline{y}_i) - f(\underline{x}_{i-1},\underline{y}_{i-1})]^2 < \epsilon \quad .$$

If this criterion is met, the point $(\underline{x}_1, \underline{y}_1)$ found is the desired solution by the definition of a saddle point -- equation (4). There remain to be shown the conditions under which the sequence of points $(\underline{x}_1, \underline{x}_2, \ldots)$ and $(\underline{y}_0, \underline{y}_1, \ldots)$ converge to $\underline{x}^*$ and $\underline{y}^*$. The Principle of Contraction Mapping [13] was applied to the class of functions considered in this thesis and conditions were found which ensure convergence for these functions (see Appendix B).

2. Simultaneous Minimaximization by Gradient Projection

Before describing this method it should be stated that it does not seem to work in general; it is included primarily to point out some of the pitfalls of using two dimensions to visualize n-dimensional concepts.

The problem of finding a saddle point of $f(\underline{x},\underline{y})$ subject to $\underline{\lambda}_x(\underline{x}) \geq \underline{0}$ and $\underline{\lambda}_y(\underline{y}) \geq \underline{0}$ was altered in the following manner. Let

$$\underline{z} = \begin{bmatrix} \underline{x} \\ \hline \underline{y} \end{bmatrix} \quad ; \quad \underline{\lambda}_z(\underline{z}) = \begin{bmatrix} \underline{\lambda}_x(\underline{x}) \\ \hline \underline{\lambda}_y(\underline{y}) \end{bmatrix}.$$

Define the "gradient" of $f(\underline{z})$ as

$$\underline{g}(\underline{z}) = \begin{bmatrix} -\nabla_{\underline{x}} f(\underline{x},\underline{y}) \\ \hline \nabla_{\underline{y}} f(\underline{x},\underline{y}) \end{bmatrix} . \tag{18}$$

It was then intended to solve the Maximum Problem

$$\max_{\underline{z}} f(\underline{z})$$

subject to

$$\underline{\lambda}_z(\underline{z}) \geq \underline{0}$$

with the "gradient" as defined in equation (18), for

$$\underline{z}^* = \begin{bmatrix} \underline{x}^* \\ \hdashline \underline{y}^* \end{bmatrix} \tag{19}$$

For the case where $\underline{x}$ and $\underline{y}$ are scalars, the progress of the algorithm may be followed on a contour plot and converges quite rapidly. Unfortunately this method does not seem to extend to n-dimensions except for isolated examples.

IV. TWO-DIMENSIONAL EXAMPLES

The methods described in the preceding section were initially tested on the following two-dimensional example. Find the saddle point of

$$f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6 \tag{20}$$

subject to

$$\begin{aligned} M_x^- &\le x \le M_x^+ \\ M_y^- &\le y \le M_y^+ \end{aligned} \tag{21}$$

The function f is convex in x and concave in y, hence, by Theorems 1 and 2 a saddle point exists. The region of x,y space for which the inequalities (21) are satisfied is called the <u>feasible region</u>. If this feasible region is large enough, the saddle point will be interior and located at

$$x^* = -1.6$$

$$y^* = 1.2 \quad .$$

Clearly, the selection of M_x^+, M_x^-, M_y^+, and M_y^- in (21) determines whether the solution will be interior or constrained.

From equations (5) through (12) necessary conditions for (x^*,y^*) to be a saddle point are

$$
\begin{aligned}
-2x^* - 2 - y^* + \mu_{x1} - \mu_{x2} &= 0 \\
(x^* + M_x^-)\,\mu_{x1} &= 0 \\
(-x^* + M_x^+)\,\mu_{x2} &= 0 \\
-2y^* - 4 + x^* + \mu_{y1} - \mu_{y2} &= 0 \\
(y^* + M_y^-)\,\mu_{y1} &= 0 \\
(-y^* + M_y^+)\,\mu_{y2} &= 0
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
x^* + M_x^- &\geq 0 & \mu_{x1} &\geq 0 \\
-x^* + M_x^+ &\geq 0\,, & \mu_{x2} &\geq 0 \\
y^* + M_y^- &\geq 0 & \mu_{y1} &\geq 0 \\
-y^* + M_y^+ &\geq 0 & \mu_{y2} &\geq 0\,.
\end{aligned}
\tag{23}
$$

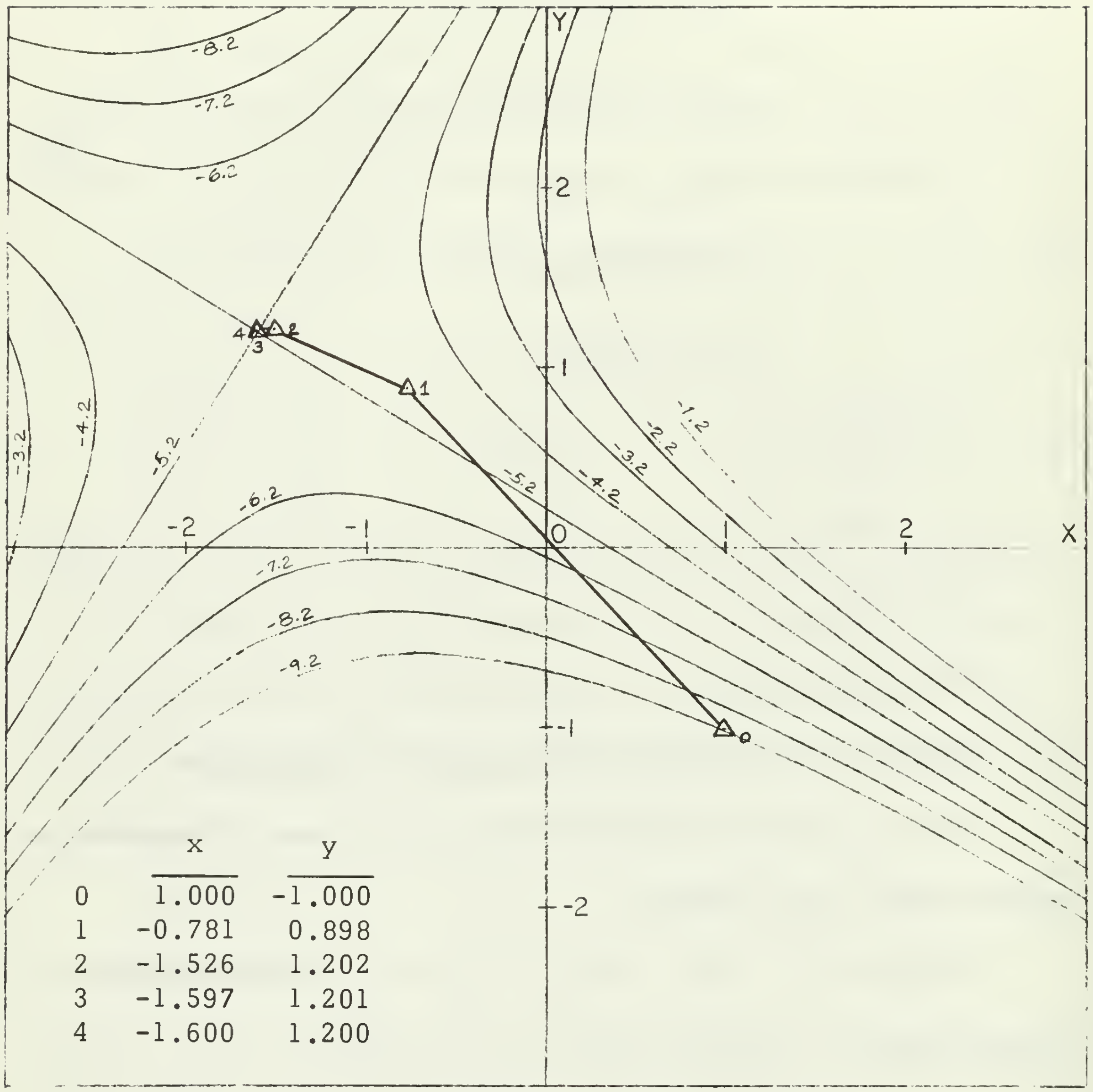
A. SOLUTION OF NONLINEAR SIMULTANEOUS EQUATIONS

All three methods converged for the several cases of the previous example which were examined.

With $M = M_x^+ = M_x^- = M_y^+ = M_y^- = 3.$, the three methods converged from the starting point (1,-1,1,1,1,1) to

$$
\begin{aligned}
x^* &= -1.6000 \\
y^* &= 1.2000 \\
\mu_{x1}, \mu_{x2}, \mu_{y1}, \mu_{y2} &= 0.0\,.
\end{aligned}
$$

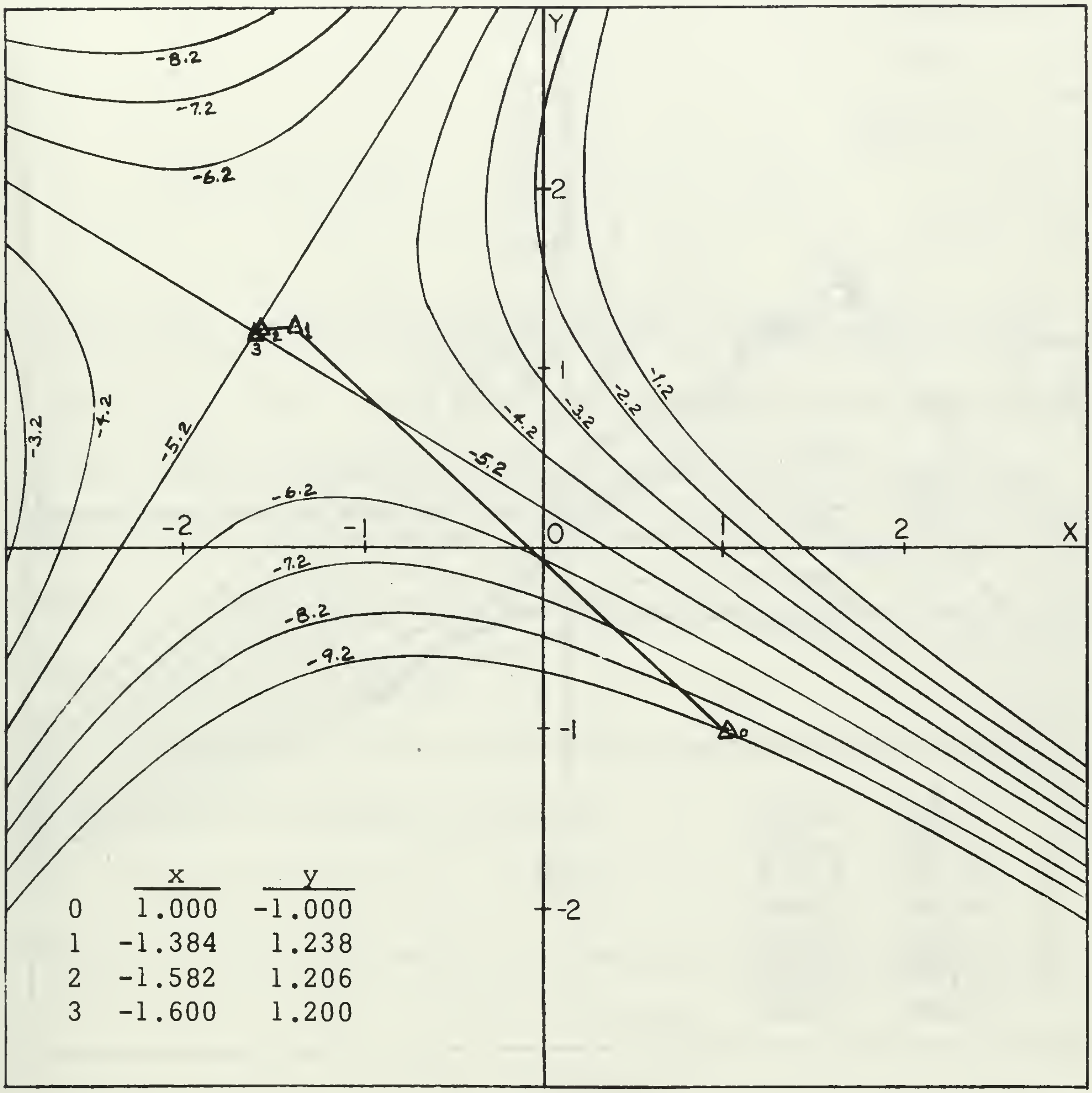
The Newton-Raphson method required four iterations, Brown's algorithm took three iterations, and the gradient projection root-finding method required five iterations. These results are shown in Figs. 1, 2, and 3 respectively.



	x	y
0	1.000	-1.000
1	-0.781	0.898
2	-1.526	1.202
3	-1.597	1.201
4	-1.600	1.200

Fig. 1

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of the Newton-Raphson method to an interior saddle point.



	x	y
0	1.000	-1.000
1	-1.384	1.238
2	-1.582	1.206
3	-1.600	1.200

Fig. 2

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress Brown's Algorithm to an interior saddle point.

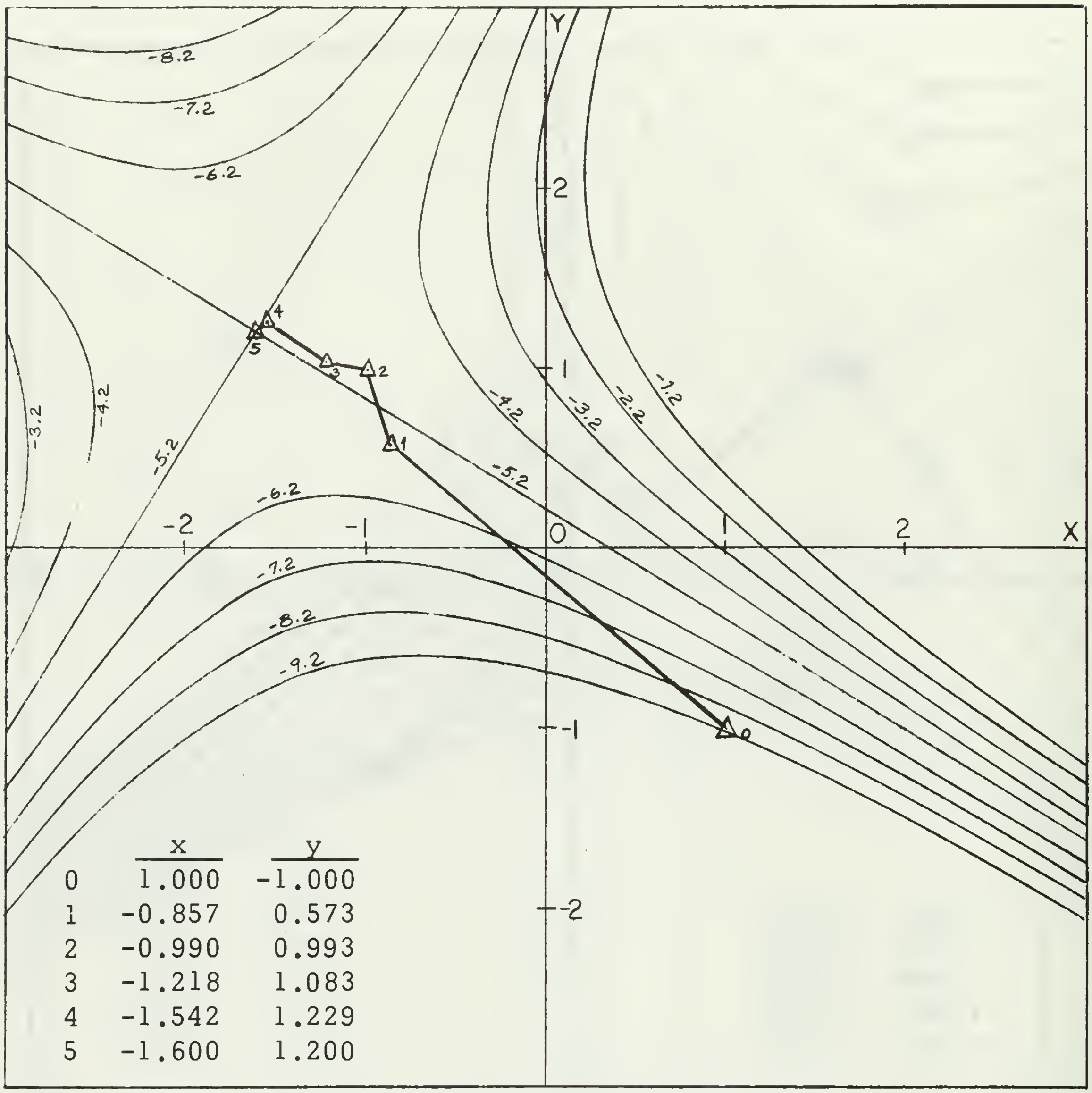


Fig. 3

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of gradient projection root-finding method to an interior saddle point.

M_y^+ was changed to 1.0 and all three methods converged to the constrained saddle point

$$x^* = -1.5000$$

$$y^* = 1.0000$$

$$\mu_{x1}, \mu_{x2}, \mu_{y1} = 0.0$$

$$\mu_{y2} = 0.5000$$

in five, four, and six iterations respectively. These results are shown in Figs. 4, 5, and 6. Note that a nonfeasible starting point was used for the gradient projection root-finding method in Fig. 6. Also note that the method does not seem to make any progress from iteration 1 to iteration 2. This is due to the fact that only two of the six variables are shown in Fig. 6.

The objective function was then changed slightly (for reasons to be explained in a following section) to

$$f(x,y) = x^2 - y^2 + 5xy + 2x + 4y - 6 \quad .$$

With M = 3., all three methods converged to the interior saddle-point

$$x^* = -0.82759$$

$$y^* = -0.06897$$

$$\mu_{x1}, \mu_{x2}, \mu_{y1}, \mu_{y2} = 0.0 \quad .$$

B. DIRECT MINIMIZATION AND MAXIMIZATION BY GRADIENT PROJECTION

1. Gradient Projection Solution of a Sequence of Constrained Constrained Optimization Problems

The function

$$f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6 \tag{24}$$

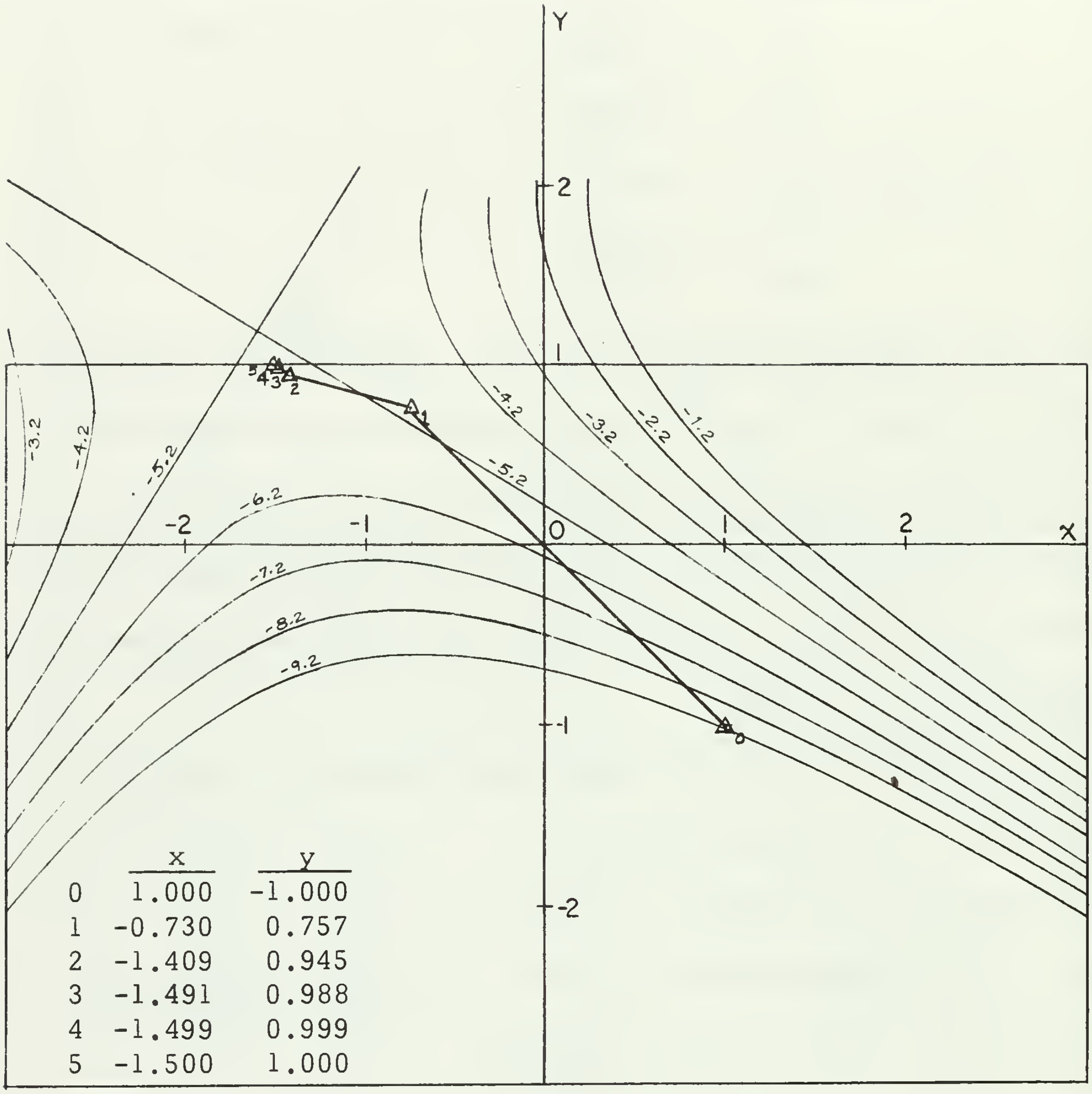


Fig. 4

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of the Newton-Raphson method to a constrained saddle point.

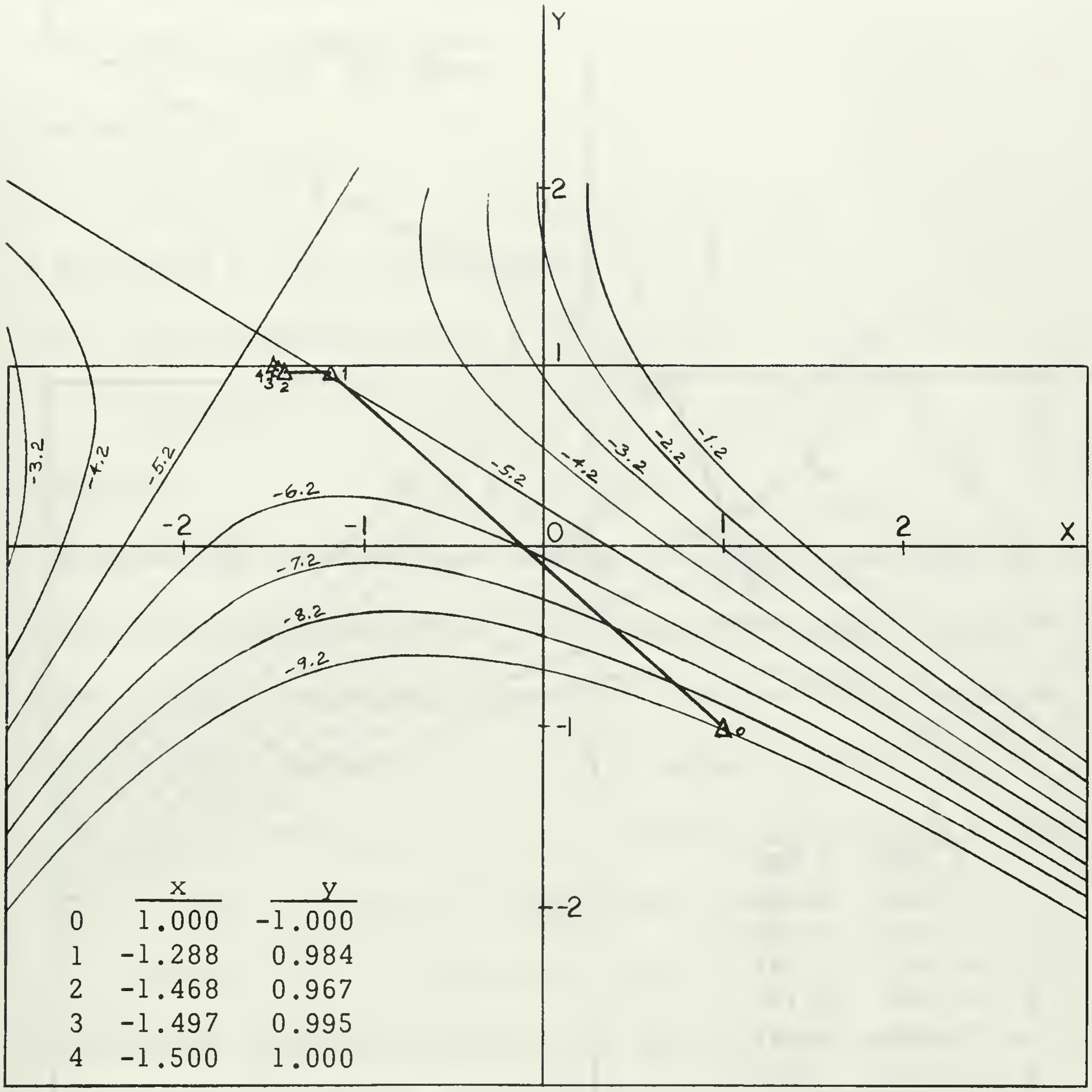
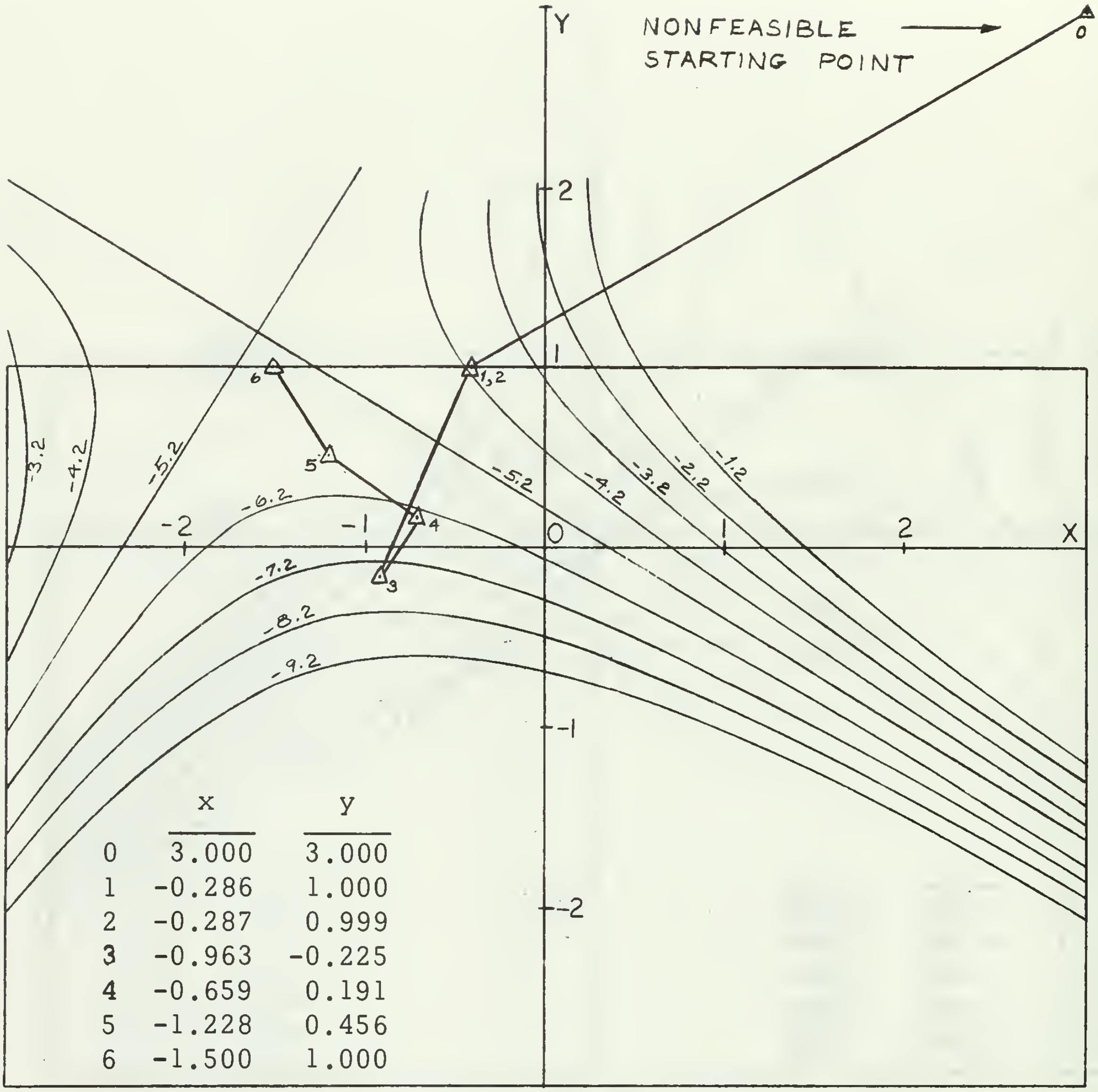


Fig. 5

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of Brown's Algorithm to a constrained saddle point.



	x	y
0	3.000	3.000
1	-0.286	1.000
2	-0.287	0.999
3	-0.963	-0.225
4	-0.659	0.191
5	-1.228	0.456
6	-1.500	1.000

Fig. 6

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of gradient projection root-finding method to a constrained saddle point.

was again used. A starting value of y was selected as 3, so the first stage of the method was to minimize

$$f(x) = x^2 + 5x - 3 \tag{25}$$

subject to

$$-3 \le x \le 3 \quad .$$

A solution of $x_1 = -2.5$ was obtained.

Then, the maximization problem

$$\max_y f(y) = -y^2 + 1.5y - 4.75 \tag{26}$$

subject to

$$-3 \le y \le 3,$$

was solved and the solution $y_2 = 0.75$ was obtained. This procedure was continued until the stopping criterion was less than 10^{-6}. The progress of the algorithm is shown in Fig. 7. The sequence of points $(x_1, x_2, \ldots\ldots\ldots)$ and $(y_1, y_2, \ldots\ldots\ldots)$ obviously converges to x^* and y^*.

Again, the objective function was change to

$$f(x,y) = x^2 - y^2 + 5xy + 2x + 4y - 6 \quad . \tag{27}$$

This time the sequence of points (x_i, y_i) (starting from the origin) diverged as shown in Fig. 8. A sufficient condition for convergence is that the equation

$$\underline{z}_i = C\underline{z}_{i-1} \tag{28}$$

be a contraction mapping (see Appendix B), where

$$\underline{z}_i = \begin{bmatrix} x_i \\ \hline y_i \end{bmatrix} \quad , \qquad C = \begin{bmatrix} 25/4 & 0 \\ 0 & 25/4 \end{bmatrix} \quad .$$

Neither of the sufficient conditions for C to be a contraction mapping are satisfied since

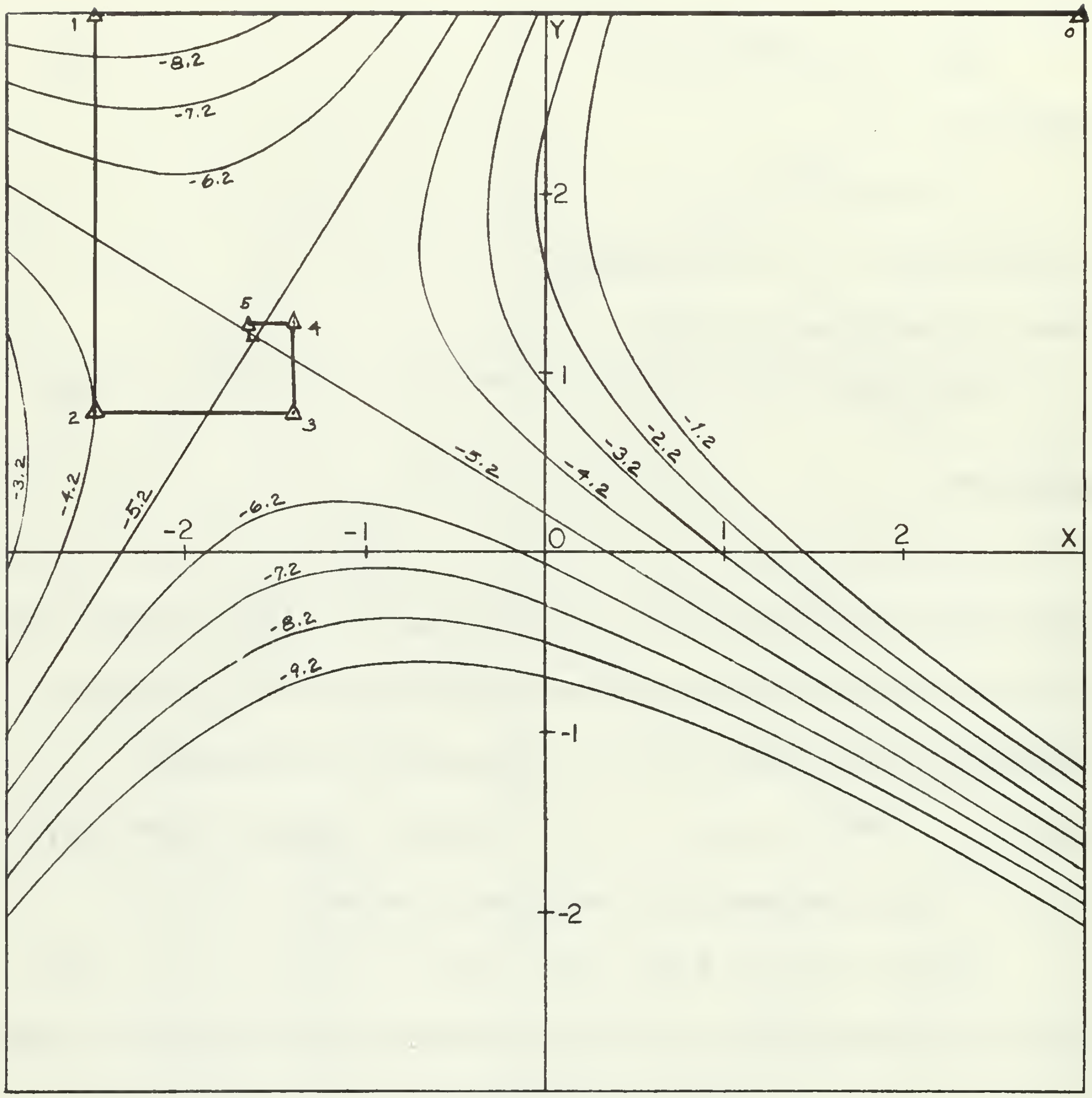


Fig. 7

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of sequential min-max by gradient projection to an interior saddle point.

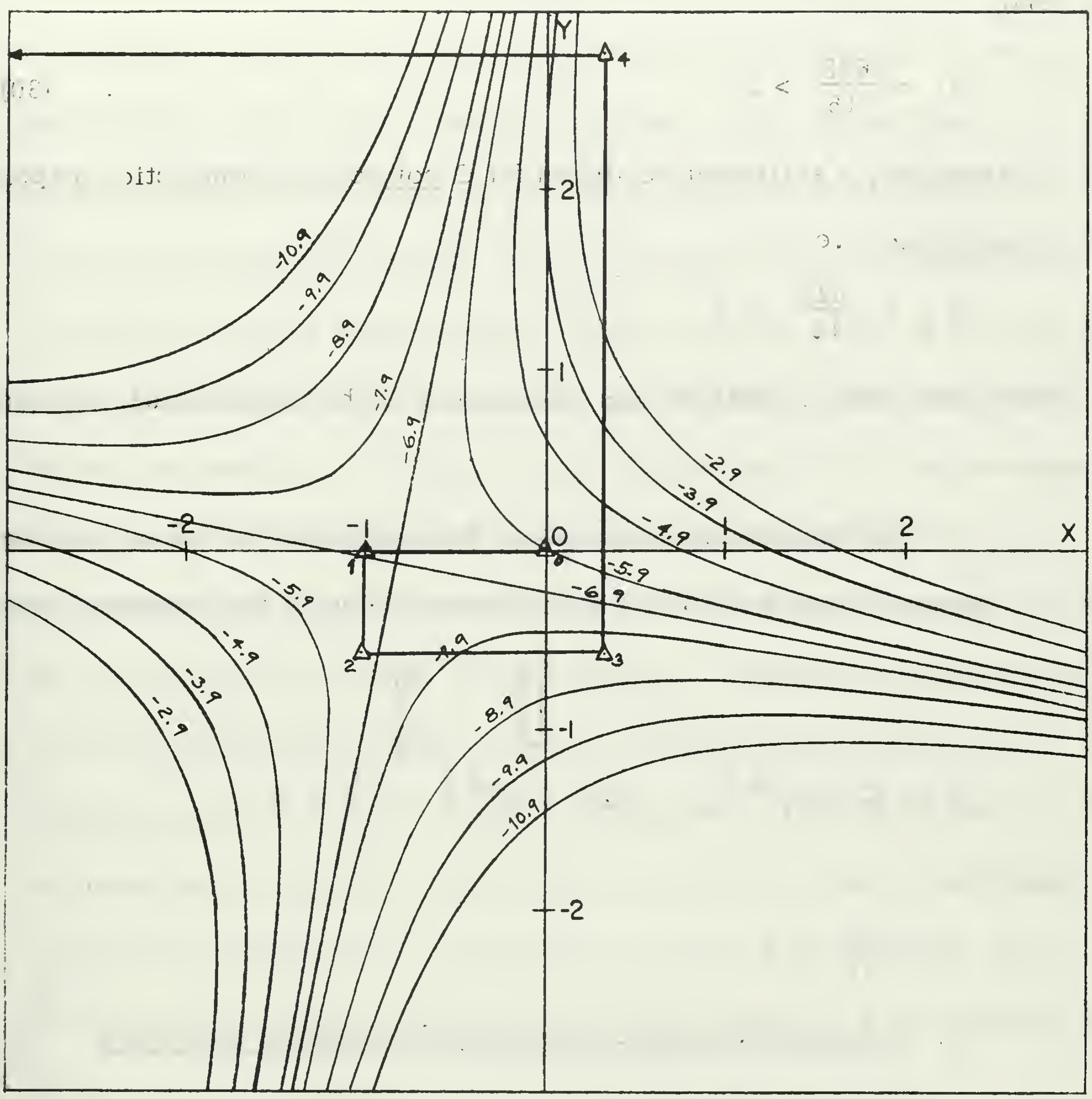


Fig. 8

Contours of $f(x,y) = x^2 - y^2 + .5xy + 2x + 4y - 6$ showing progress of sequential min-max by gradient projection.

$$\left\{\Sigma_i \, \Sigma_j (a_{ij})^2\right\} = \left\{\left(\frac{25}{4}\right)^2 + \left(\frac{25}{4}\right)^2\right\} = \frac{1250}{16} > 1 \qquad (29)$$

and

$$\lambda_1 = \frac{625}{16} > 1 \qquad (30)$$

Furthermore, a sufficient condition for C not to be a contraction mapping is satisfied, i.e.,

$$\lambda_N = \frac{625}{16} > 1 \quad .$$

Note that these conditions are independent of the initial values of x and y.

For the previous case where the coefficient of the xy term was 1, the conditions for C to be a contraction mapping are satisfied, since

$$C = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} ,$$

$$\left\{\Sigma_i \, \Sigma_j (a_{ij})^2\right\} = \left\{(\tfrac{1}{4})^2 + (\tfrac{1}{4})^2\right\} = \frac{1}{8} < 1$$

and

$$\lambda_1 = \frac{1}{16} < 1 \quad .$$

2. Simultaneous Minimaximization by Gradient Projection

Again, the function

$$f(x, y) = x^2 - y^2 + xy + 2x + 4y - 6 \qquad (31)$$

subject to

$$-3 \leq x \leq 3.$$

$$-3 \leq y \leq 3.$$

is used to illustrated the method.

The starting point $x_0 = 1$, $y_0 = 1$ was selected, and the negative gradient with respect to x and the positive gradient with respect to y were computed. These two components form the "plus-minus gradient" defined in equation (18). They are shown by heavy lines in Fig. 9. The true gradient and it's components are shown by dotted lines. A step is taken in this "plus-minus" gradient direction. The gradient projection algorithm takes the maximum step in the direction of the gradient (in this case the plus-minus gradient), i.e., it moves as far as possible without leaving the bounded search region. In this case the point $1'$ (-1.6,3.), which lies on the constraint boundary, is reached. The algorithm then determines whether or not this maximum step is too far by computing the inner product of the normalized projected gradient at point 0 and the gradient at point $1'$. If this inner product is greater than or equal to zero the maximum step is taken. If it is negative, repeated linear interpolation is used to determine the point where the inner product is zero, i.e., where the gradient is orthogonal to the normalized projected gradient evaluated at the previous point. In this case, the maximum step is too far and by repeated linear interpolation the point 1 is obtained. This procedure was continued until the necessary conditions for convergence of the algorithm were met.

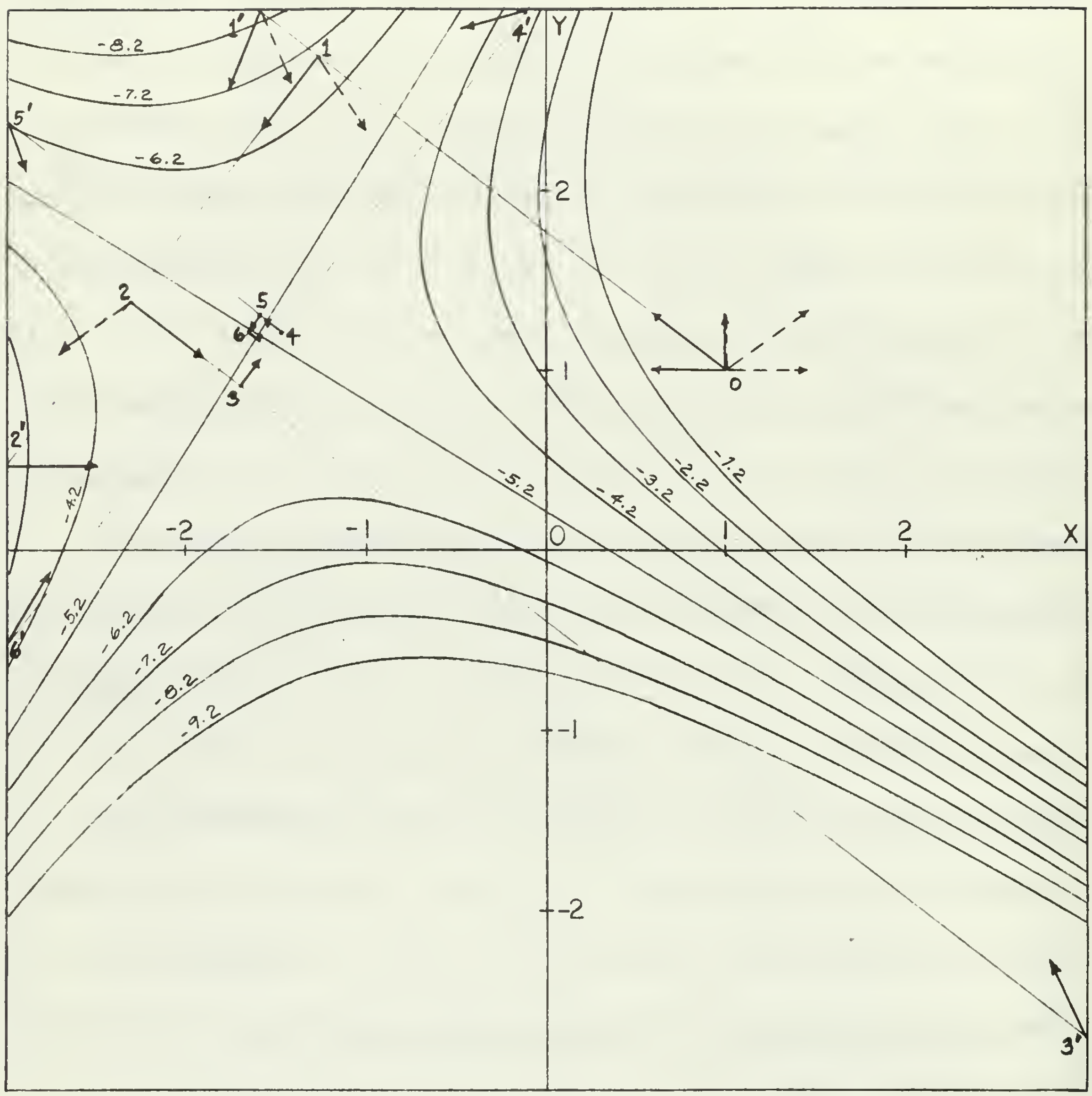


Fig. 9

Contours of $f(x,y) = x^2 - y^2 + xy + 2x + 4y - 6$ showing progress of simultaneous min-max by gradient projection to an interior saddle point.

V. A SPECIAL CASE

In his investigation of two-sided allocation problems, Bartley [1] proposed a solution for a special type of saddle-point problem.

A functional of the form

$$I = \int_a^b K \, dt \tag{32}$$

is considered where $K = K(x, y, t)$ is a real valued function. It is desired to find a function x which minimizes (32) subject to the constraints

$$x \geq 0 \qquad \text{and} \qquad X = \int_a^b x \, dt = 1 \tag{33}$$

and a function y which maximizes (13) subject to

$$y \geq 0 \qquad \text{and} \qquad Y = \int_a^b y \, dt = 1 \tag{34}$$

i.e., a saddle point of the functional I is sought subject to the constraints (33) and (34).

It is convenient to introduce a new functional

$$J = \int_a^b H \, dt = \int_a^b (D + \mu_x x + \mu_y y) \, dt \tag{35}$$

where μ_x and μ_y are constants (Lagrange multiplier's) to be found. Note that if x and y are admissible (equations (33) and (34) are satisfied) then J differs from I by the amount $\mu_x + \mu_y$.

Now if the functions x° and y° satisfy

$$J(x^\circ, y) \leq J(x^\circ, y^\circ) \leq J(x, y^\circ) \tag{36}$$

they are a saddle-point of J. But they may not be admissible functions. If they are admissible <u>and</u> satisfy (36) they are called the min-max strategies, x^* and y^*.

Bartley proposed that μ_x and μ_y be selected arbitrarily. Then the Kuhn-Tucker necessary conditions for a saddle-point [14] can be applied to the integrand, H, of (35). These conditions are

$$\begin{aligned} &\frac{\partial H}{\partial x}(x^\circ, y^\circ) \geq 0 \quad , \quad \left[\frac{\partial H}{\partial x}(x^\circ, y^\circ)\right] x^\circ = 0 \quad , \quad x^\circ \geq 0 \\ &\frac{\partial H}{\partial y}(x^\circ, y^\circ) \leq 0 \quad , \quad \left[\frac{\partial H}{\partial y}(x^\circ, y^\circ)\right] y^\circ = 0 \quad , \quad y^\circ \geq 0 \end{aligned} \tag{37}$$

for all $t \in [a,b]$, can be used to obtain a trial solution (x°, y°). If this trial solution is admissible, then the min-max strategies have been found. If the trial solution is not admissible, Bartley used a specially tailored Newton-Raphson iteration to correct the value of (X, Y) and obtain a sequence approaching (1,1).

A particular functional considered by Bartley was,

$$J(x, y) = \int_0^1 [\,(x+2t)^2 + xyt - (y+3t)^2\,]\, dt \tag{38}$$

By dividing the interval [0,1] into N subintervals each of length T and assuming piecewise-constant x and y over T, the integrals can be approximated by summations. (Note the if the problem is to be solved on the computer there is a discretization performed eventually anyway).

Now the problem is in the form: Find a saddle point of

$$J_D = T \sum_{k=0}^{N-1} [\,(x(k) + 2\frac{k}{N})^2 + x(k)y(k)\frac{k}{N} - (y(k) + 3\frac{k}{N})^2\,] \tag{39}$$

subject to

$$\begin{aligned} x(k) &\geq 0 \quad , \quad k = 0, \ldots, N-1 \\ y(k) &\geq 0 \quad , \quad k = 0, \ldots, N-1 \end{aligned} \tag{40}$$

and

$$T \sum_{k=0}^{N-1} x(k) = 1$$
$$T \sum_{k=0}^{N-1} y(k) = 1 \tag{41}$$

where the saddle point of J_D, which is a function of 2N variables subject to 2N+4 constraints (if the equalities are written as two inequalities as in equation (15)), can be found by the methods described previously.

This example was solved by the sequential min-max gradient projection method with N=25 and gave the trajectories shown in Fig. 10. These correspond quite closely to Bartley's results, also shown in Fig. 10.

The simultaneous min-max method was also tried on this problem and gave the same results as the sequential method. Although, as mentioned previously, the simultaneous min-max method does not seem to extend in general to examples of dimension greater than second-order, the feasible region in this case was apparently so restricted by the equalities that the algorithm was able to converge.

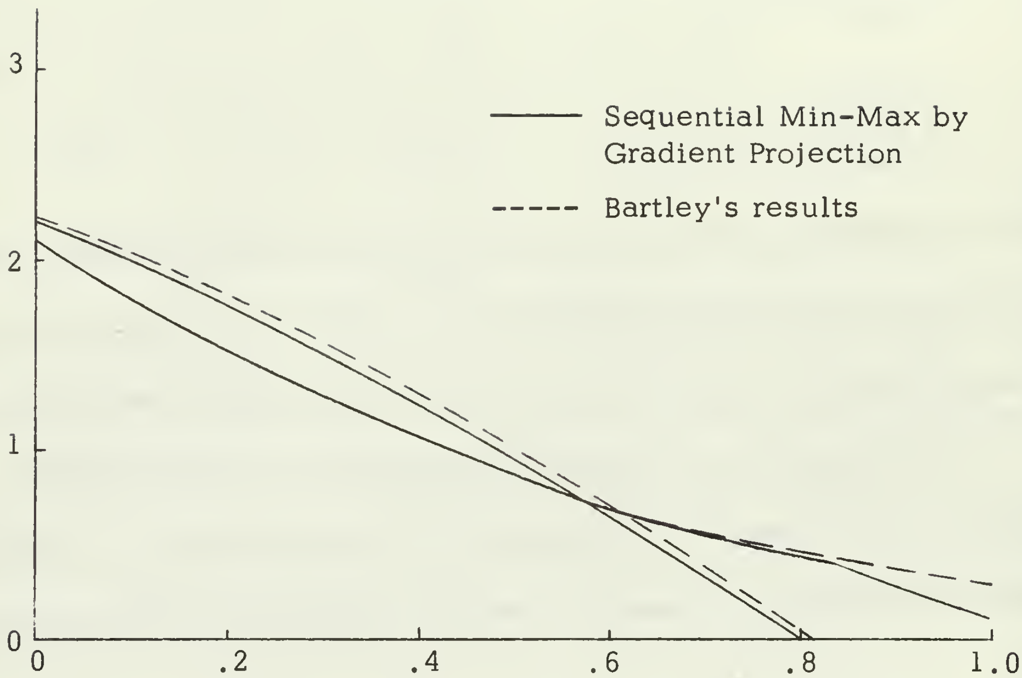


Fig. 10

Functions x and y which are a saddle point of

$$J = \int_0^1 [\,(x + 2t)^2 + xyt - (y + 3t)^2\,]\, dt$$

subject to $\int_0^1 x\, dt = 1,\ x \geq 0,\quad \int_0^1 y\, dt = 1,\ y \geq 0.$

VI. A PURSUIT-EVASION PROBLEM

A pursuit-evasion problem was also considered in which each player's dynamic system is described by linear differential equations and the payoff is a weighted combination of terminal miss distance and each player's control effort expended.

The state equations of two systems are

$$\underline{x}_p(t) = A_p(t)\underline{x}_p(t) + B_p(t)\underline{u}_p(t) \; ; \; \underline{x}_p(t_0) = \underline{x}_{p0}$$

$$\underline{x}_e(t) = A_e(t)\underline{x}_e(t) + B_e(t)\underline{u}_e(t) \; ; \; \underline{x}_e(t_0) = \underline{x}_{e0} \qquad (42)$$

$$\underline{u}_p(t), \; \underline{u}_e(t) \in E^q$$

where $\underline{x}_p$ and $\underline{x}_e$ are n-vectors describing the respective players' states at any time t, $\underline{u}_p$ and $\underline{u}_e$ are q-dimensional control vectors for each player,[7] and E^q is q-dimensional Euclidean space. The problem is to find a saddle point $(\underline{u}_p^*, \underline{u}_e^*)$ of the payoff

$$J = [\underline{x}_p(t_f) - \underline{x}_e(t_f)]^T \; Q \; [\underline{x}_p(t_f) - \underline{x}_e(t_f)]$$
$$+ \int_{t_0}^{t_f} [\underline{u}_p(t)^T R_p \underline{u}_p(t) - \underline{u}_e(t)^T R_e \underline{u}_e(t)] \, dt \qquad (43)$$

where Q is a positive semidefinite matrix and R_p and R_e are positive definite matrices. This is essentially the problem solved by Ho, Bryson, and Baron [6]. The solution is well known and has been presented by many authors, including Hutchison and Permenter [7], and Willman [20].

---

[7] The dimension of each player's control vector may be different although for simplicity they are assumed equal here.

Additional constraints on the control variables are assumed, such as

$$|u_{pi}(t)| \leq M_p \quad , \qquad i = 1,\ldots,q \tag{44}$$

$$|u_{ei}(t)| \leq M_e \quad , \qquad i = 1,\ldots,q \quad . \tag{45}$$

Any other equality or inequality constraints on either the state or control variables may also be included (Jacob and Polak [10] include energy constraints), but for simplicity only the above are considered. The calculus of variations as an analytical tool is no longer as attractive as it was in the unconstrained problem. Although Berkowitz, in his definitive paper based on the calculus of variations [3] , considered constraints theoretically, apparently variational techniques have not been applied to solve meaningful examples with constraints. This deficiency is important because it is quite realistic to expect constraints on the control variables of physical systems, e.g., aircraft do not have unlimited control surface deflection nor unlimited acceleration. Furthermore, in a pursuit-evasion situation it is realistic to expect control variables to always be at their limits, e.g., an aircraft commander will use every bit of maneuverability built into his aircraft to avoid a pursuing missile.

The procedure used to solve this problem was to assume that the control variables are piecewise constant, and to approximate the differential equations by difference equations and the integral in the payoff functional by a summation. Then the nonlinear programming techniques discussed previously were applied.

If the control variables are piecewise constant, the state equations can be written as

$$\underline{x}_p(k+1) = \Phi_p\left((k+1)T,kT\right)\underline{x}_p(k) + \Gamma_p\left((k+1)T,kT\right)\underline{u}_p(k)$$
$$\underline{x}_e(k+1) = \Phi_e\left((k+1)T,kT\right)\underline{x}_e(k) + \Gamma_e\left((k+1)T,kT\right)\underline{u}_e(k) \tag{46}$$

where $\Phi_p((k+1)T,kT)$ and $\Phi_e((k+1)T,kT)$ are the respective state transition matrices [12], and $\Gamma_p((k+1)T,kT)$, $\Gamma_e((k+1)T, kT)$ are given by

$$\Gamma_p((k+1)T,kT) = \int_{kT}^{(k+1)T} \Phi_p((k+1)T,t)B_p(t)\,dt \tag{47}$$

$$\Gamma_e((k+1)T,kT) = \int_{kT}^{(k+1)T} \Phi_e((k+1)T,t)B_e(t)\,dt \tag{48}$$

respectively. Although the following development can be performed for time-varying linear systems, for notational simplicity it is assumed that $A_p(t)$, $A_e(t)$, $B_p(t)$, and $B_e(t)$ are constant matrices in which case

$$\Phi_p((k+1)T,kT) = \Phi_p(T)$$

$$\Phi_e((k+1)T,kT) = \Phi_e(T)$$

$$\Gamma_p((k+1)T,kT) = \Gamma_p(T)$$

$$\Gamma_e((k+1)T,kT) = \Gamma_e(T)$$

A discrete approximation to J can now be written as

$$J_D = [\underline{x}_p(N)-\underline{x}_e(N)]^T Q\ [\underline{x}_p(N) - \underline{x}_e(N)]$$
$$+ T \sum_{k=0}^{N-1} [\underline{u}_p(k)^T R_p\, \underline{u}_p(k) - \underline{u}_e(k)^T R_e\, \underline{u}_e(k)\ ] . \tag{49}$$

Writing out a few terms for the pursuer's system gives

$$\underline{x}_p(1) = \Phi_p(T)\underline{x}_p(0) + \Gamma_p(T)\underline{u}_p(0)$$

$$\begin{aligned}
\underline{x}_p(2) &= \Phi_p(T)\underline{x}_p(1) + \Gamma_p(T)\underline{u}_p(1) \\
&= \Phi_p(T)\,[\Phi_p(T)\underline{x}_p(0) + \Gamma_p(T)\underline{u}_p(0)] + \Gamma_p(T)\underline{u}_p(1) \\
&= \Phi_p^2(T)\underline{x}_p(0) + \Phi_p(T)\Gamma_p(T)\underline{u}_p(0) + \Gamma_p(T)\underline{u}_p(1)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\underline{x}_p(N) = {} & \Phi_p^N(T)\underline{x}_p(0) + \Phi_p^{N-1}(T)\Gamma_p(T)\underline{u}_p(0) + \Phi_p^{N-2}(T)\Gamma_p(T)\underline{u}_p(1) \\
& + \ldots + \Phi_p(T)\,\Gamma_p(T)\underline{u}_p(N-2) + \Gamma_p(T)\underline{u}_p(N-1) \quad . \qquad (50)
\end{aligned}$$

A similar expression can be written for $\underline{x}_e(N)$. Thus it is possible to express the final state values in terms of the initial state and the N values of the control vector. Substituting equation (50) into equation (49) allows the payoff to be written as a function of the 2Nq control values

$$\bar{\underline{u}}_p \equiv \begin{bmatrix} \underline{u}_p(0) \\ \hline \underline{u}_p(1) \\ \hline \vdots \\ \hline \underline{u}_p(N-1) \end{bmatrix} , \quad \bar{\underline{u}}_e \equiv \begin{bmatrix} \underline{u}_e(0) \\ \hline \underline{u}_e(1) \\ \hline \vdots \\ \hline \underline{u}_e(N-1) \end{bmatrix} . \qquad (51)$$

The nxNq matrices $\bar{\Phi}_p$ and $\bar{\Phi}_e$ are defined as

$$\bar{\Phi}_p \equiv \left[\Phi_p^{N-1}(T)\Gamma_p(T) \;\vdots\; \Phi_p^{N-2}(T)\Gamma_p(T) \;\vdots\; --- \;\vdots\; \Gamma_p(T) \right] , \qquad (52)$$

$$\bar{\Phi}_e \equiv \left[\Phi_e^{N-1}(T)\Gamma_e(T) \;\vdots\; \Phi_e^{N-2}(T)\Gamma_e(T) \;\vdots\; --- \;\vdots\; \Gamma_e(T) \right] , \qquad (53)$$

the NqxNq matrices $\bar{R}_p$ and $\bar{R}_e$ are given by

$$\bar{R}_p \equiv \begin{bmatrix} R_p & 0 & \cdots & 0 \\ 0 & R_p & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & R_p \end{bmatrix} , \tag{54}$$

$$\bar{R}_e \equiv \begin{bmatrix} R_e & 0 & \cdots & 0 \\ 0 & R_e & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & R_e \end{bmatrix} , \tag{55}$$

and

$$\theta \equiv \Phi_p^N(T)\underline{x}_p(0) - \Phi_e^N(T)\underline{x}_e(0) , \tag{56}$$

where θ is the "predicted terminal miss" at time t=NT if no controls are applied during the preceding N intervals.

J_D can now be written as

$$J_D = [\theta + \bar{\Phi}_p\underline{\bar{u}}_p - \bar{\Phi}_e\underline{\bar{u}}_e]^T Q[\theta + \bar{\Phi}_p\underline{\bar{u}}_p - \bar{\Phi}_e\underline{\bar{u}}_e] + T[\underline{\bar{u}}_p^T\bar{R}_p\underline{\bar{u}}_p - \underline{\bar{u}}_e^T\bar{R}_e\underline{\bar{u}}_e]$$

$$= \underline{\bar{u}}_p^T[\bar{\Phi}_p^T Q\bar{\Phi}_p + \bar{R}_p]\underline{\bar{u}}_p + \underline{\bar{u}}_e^T[\bar{\Phi}_e^T Q\bar{\Phi}_e - \bar{R}_e]\underline{\bar{u}}_e$$

$$+ 2\left\{\underline{\bar{u}}_p^T[\bar{\Phi}_p^T Q\theta] - \underline{\bar{u}}_p[\bar{\Phi}_p^T Q\bar{\Phi}_e]\underline{\bar{u}}_e - \underline{\bar{u}}_e^T[\bar{\Phi}_e^T Q\theta]\right\} + \theta^T Q\theta. \tag{57}$$

Since it was assumed that Q is positive semidefinite and R_p positive definite, it follows that the matrix $[\bar{\Phi}_p^T Q\bar{\Phi}_p + \bar{R}_p]$ is positive definite and hence that J_D is a convex function of $\underline{\bar{u}}_p$. If J_D is also a concave function of $\underline{\bar{u}}_e$, then J_D will have a saddle point. This will be the case if the matrix $\bar{R}_e$ is weighted relative to $\bar{\Phi}_e^T Q \bar{\Phi}_e$ in such a way that $[\bar{\Phi}_e^T Q \bar{\Phi}_e - \bar{R}_e]$ is negative semidefinite.

Assuming that $[\bar{\Phi}_e^T Q \bar{\Phi}_e - \bar{R}_e]$ is negative semidefinite, the previously described methods can be applied. The constraints (44) and (45) are written as

$$\underline{\lambda}_p(\bar{\underline{u}}_p) = \begin{bmatrix} I_q \\ \hline -I_q \end{bmatrix} \bar{\underline{u}}_p - (-\underline{m}_p) \geq \underline{0} \tag{58}$$

$$\underline{\lambda}_e(\bar{\underline{u}}_e) = \begin{bmatrix} I_q \\ \hline -I_q \end{bmatrix} \bar{\underline{u}}_e - (-\underline{m}_e) \geq \underline{0} \tag{59}$$

where I_q is the q-dimensional identity matrix, $\underline{m}_p$ is the 2q-dimensional vector of constants M_p, $\underline{m}_e$ is the 2q-dimensional vector of constants M_e, and the problem may be stated in the concise form: Determine a saddle point of (57) subject to the constraints (58) and (59).

For $(\bar{\underline{u}}_p^*, \bar{\underline{u}}_e^*)$ to be a solution to this problem, it is necessary and sufficient that $(\bar{\underline{u}}_p^*, \bar{\underline{u}}_e^*)$ and the 2 q-dimensional vectors $\underline{\mu}_p$ and $\underline{\mu}_e$ satisfy

$$\begin{aligned} \nabla_{\bar{\underline{u}}_p} J_D(\bar{\underline{u}}_p^*, \bar{\underline{u}}_e^*) + [I_q \mid -I_q] \underline{\mu}_p &= \underline{0} \\ \underline{\lambda}_p(\bar{\underline{u}}_p^*)^T \underline{\mu}_p &= 0 \\ \underline{\lambda}_p(\bar{\underline{u}}_p^*) &\geq \underline{0} \\ \underline{\mu}_p &\geq \underline{0} \end{aligned}$$

$$-\nabla_{\bar{\underline{u}}_e} J_D(\bar{\underline{u}}_p^*, \bar{\underline{u}}_e^*) + [I_q \mid -I_q]\, \underline{\mu}_e = \underline{0} \tag{60}$$

$$\underline{\lambda}_e(\bar{\underline{u}}_e^*)^T \underline{\mu}_e = 0$$

$$\underline{\lambda}_e(\bar{\underline{u}}_e^*) \geq \underline{0}$$

$$\underline{\mu}_e \geq \underline{0}$$

The following simplified pursuit-evasion game was solved by using a Newton-Raphson method, Brown's algorithm, and by sequential minimization and maximization using the gradient projection algorithm.

Let

$$A = A_p(t) = A_e(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = B_p(t) = B_e(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R_p = \begin{bmatrix} r_p & 0 \\ 0 & r_p \end{bmatrix}; \quad R_e = \begin{bmatrix} r_e & 0 \\ 0 & r_e \end{bmatrix}$$

$$\underline{x}_p(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \underline{x}_e(0) = \begin{bmatrix} 10 \\ -2 \end{bmatrix}.$$

The initial time was set at 0, the final time at 2, and the sample interval was taken as T=1.

The state transition matrix is

$$\Phi = \Phi_p(T) = \Phi_e(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and the matrix Γ is given by

$$\Gamma = \Gamma_p(T) = \Gamma_e(T) = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} .$$

J_D becomes

$$\begin{aligned} J_D = \bar{\underline{u}}_p^T & \left\{ \begin{bmatrix} 9/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix} + \begin{bmatrix} r_p & 0 \\ 0 & r_p \end{bmatrix} \right\} \bar{\underline{u}}_p \\ & + \bar{\underline{u}}_e^T \left\{ \begin{bmatrix} 9/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix} - \begin{bmatrix} r_e & 0 \\ 0 & r_e \end{bmatrix} \right\} \bar{\underline{u}}_e \\ & + 2 \left\{ \bar{\underline{u}}_p^T \begin{bmatrix} -3 \\ -1 \end{bmatrix} - \bar{\underline{u}}_p^T \begin{bmatrix} 9/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix} \bar{\underline{u}}_e - \bar{\underline{u}}_e^T \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\} + 4. \end{aligned} \tag{61}$$

and for J_D to have a saddle point it is sufficient that

$$r_p > 0$$

$$r_e > 9/4 .$$

Note that J_D is a convex function of $\bar{\underline{u}}_p$ for $r_p > -\frac{1}{4}$ but since R_p was assumed positive definite it is necessary that $r_p > 0$.

Values of $r_p = 1.$ and $r_e = 5.$ were initially chosen and with $M_p = M_e = 2.$, the Newton-Raphson method, Brown's algorithm and sequential min-max with gradient projection gave the interior saddle-point

$$u_p(0) = 1.000 \qquad u_e(0) = 0.200$$

$$u_p(1) = 0.333 \quad , \quad u_e(1) = 0.067 .$$

The Newtón-Raphson method and Brown's algorithm both converged in four iterations while the gradient projection method required 24 minimizations and maximizations.

The value of M_p was then changed to M_p = =0.8. In six iterations, the sequential min-max method converged to

$$u_p(0) = 0.800 \qquad u_e(0) = 0.320$$
,
$$u_p(1) = 0.533 \qquad u_e(1) = 0.107 \quad .$$

The Newton-Raphson method required four iterations to converge to the same point. Depending on the starting point, Brown's algorithm converged to the unconstrained solution or created a singularity in the vicinity of the constrained solution, but it did not converge to the constrained solution.

VII. CONCLUSIONS

Nonlinear programming methods have been applied to the solution of constrained saddle-point problems, in particular, to a type of problem arising in pursuit-evasion games with inequality constraints on the values of the control variables. The payoff functions considered were real-valued, concave-convex quadratic polynomials, although nonlinear programming methods are not, in general, restricted to such functions.

Five techniques were suggested - - the two most promising being the Newton-Raphson method to solve nonlinear simultaneous equations obtained from the Kuhn-Tucker necessary conditions, and the solution of a sequence of minimization-maximization problems using the gradient projection algorithm. The Newton-Raphson method was found to be somewhat sensitive to the starting point chosen. Gradient projection, while insensitive to the starting point (even a nonfeasible point is acceptable), was found to converge only when certain conditions on the function were satisfied. These conditions were found by an application of the Contraction Mapping Principle.

The primary advantage of these methods is that they admit inequality constraints on the variables, in fact, it is required that the variables be constrained.

A serious disadvantage (at least in the case of pursuit-evasion problems) is that only an open-loop strategy is obtained. A pursuit-evasion problem with inequality constraints for which a closed-loop, or feedback, strategy is desired must (apparently) be approached using calculus of variations techniques [4].

In addition to the inclusion of inequality constraints, differential game theory has many unanswered questions. Perhaps the most obvious flaw in the formulation of pursuit-evasion games is the lack of realism in the performance criterion. It has been pointed out [7] that an aircraft commander being pursued by a missile will be unsatisfied with a strategy which tells him to conserve his energy, yet it was found that for a saddle-point to exist the evader's control energy term must be weighted rather heavily in comparison to the terminal-miss term. This raises a very fundamental question concerning what is meant by "optimal" in optimal strategies. In the formulation given in this thesis, optimal means mini-max strategies which are a saddle point of the performance criterion. But, just as in optimal control theory, the engineer or analyst must be able to formulate realistic performance criteria if the "optimal" control, or strategy, is to have physical meaning.

A more realistic payoff for pursuit-evasion games would be simply a measure of the terminal miss, but a saddle point may not exist for this criterion and the meaning of the term "solution" must be altered. Very little information is available on differential games for which a saddle point does not exist. It is well known that matrix games that do not have pure strategy solutions (no saddle point exists) have solutions in mixed strategies. A strategy is called "mixed" if it is chosen from a set of admissible strategies in accordance with some particular probability distribution. Berkowitz [4] discusses the notion of mixed strategies for differential games but observes that the mathematical difficulties which arise have not been overcome.

Another approach which warrants investigation for possible application to differential game theory is introduced by Salmon [18]. Although limiting his attention to the design of controllers for systems with uncertain parameters, Salmon proposes an algorithm to solve a class of minimax problems for which a saddle point need not exist.

Further investigation of the methods suggested in this thesis should include the solution of meaningful examples, e.g., a pursuit-evasion problem with enough samples over the time interval to adequately represent the trajectories. It is expected that the conditions for convergence of the sequential min-max method would prove somewhat restrictive. Difficulty with convergence of Brown's algorithm has already been experienced and could be expected with the Newton-Raphson method in larger dimension systems.

APPENDIX A

In this appendix it is shown that Theorem 4 of Rosen's gradient projection algorithm [15] is equivalent to the Kuhn-Tucker necessary conditions for the nonlinear programming problem.

Kuhn and Tucker stated the Maximum Problem as

$$\max f(\underline{x}) \quad , \quad \underline{x} \text{ an m-vector}$$

$$\text{subject to } \underline{\lambda}(\underline{x}) \geq \underline{0} \quad , \quad \underline{\lambda} \text{ a k-vector}$$

$$\underline{x} \geq \underline{0} \quad .$$

The necessary conditions for $\underline{x}^*$ to be a solution to this problem are that $\underline{x}^*$ and some $\underline{\mu}$ satisfy:

$$\nabla_{\underline{x}} F(\underline{x}^*) = \nabla_{\underline{x}} f(\underline{x}^*) + \sum_{i=1}^{k} [\mu_i \nabla_{\underline{x}} \lambda_i(\underline{x}^*)] \leq \underline{0} \tag{A1}$$

$$\left\{ \nabla_{\underline{x}} f(\underline{x}^*) + \sum_{i=1}^{k} [\mu_i \nabla_{\underline{x}} \lambda_i(\underline{x}^*)] \right\}^T \underline{x}^* = 0 \tag{A2}$$

$$\underline{x}^* \geq \underline{0} \tag{A3}$$

$$\underline{\lambda}(\underline{x}^*) \geq \underline{0} \tag{A4}$$

$$\underline{\lambda}(\underline{x}^*)^T \underline{\mu} = 0 \tag{A5}$$

$$\underline{\mu} \geq \underline{0} \tag{A6}$$

where $F(\underline{x})$ is the Lagrangian function, $F(\underline{x})=f(\underline{x}) + \underline{\mu}^T\underline{\lambda}(\underline{x})$ and the constraints $\underline{\lambda}(\underline{x}) \geq \underline{0}$, $\underline{x} \geq \underline{0}$ must satisfy the constraint qualification[14] . Rosen [15] considered only linear constraints in Part I of his algorithm, and linear constraints always satisfy the constraint qualification.

The linear constraints can always be written in the form

$$\lambda_i(\underline{x}) = \underline{x}^T \underline{n}_i - b_i \geq 0 \quad , \quad i=1,\ldots k \tag{A7}$$

where the $\underline{n}_i$ have been normalized, that is,

$$\sum_{j=1}^{m} (n_{ij})^2 = 1 \quad , \quad i=1,\ldots,k \quad . \tag{A8}$$

Note that

$$\nabla_{\underline{x}} \lambda_i(\underline{x}) = \underline{n}_i \quad . \tag{A9}$$

The mxk matrix N_k is defined as

$$N_k = [\underline{n}_1 \, \underline{n}_2 \quad . \; . \; . \quad n_k] \tag{A10}$$

(for linear constraints this corresponds to F° in [14]), and the k-vector $\underline{b}$ as

$$\underline{b} = [b_1, b_2, \ldots , n_k]^T \quad . \tag{A11}$$

Using these definitions, the system of k inequalities can be written as

$$N_k^T \; \underline{x} - \underline{b} \geq \underline{0} \quad . \tag{A12}$$

In addition Rosen does not require that $\underline{x} \geq \underline{0}$; therefore, (A1) must hold as an equality, and the necessary conditions become

$$\begin{aligned} \nabla_{\underline{x}} f(\underline{x}^*) + N_k \, \underline{\mu} &= \underline{0} \\ N_k^T \; \underline{x}^* - \underline{b} &\geq \underline{0} \\ [N_k^T \; \underline{x}^* - \underline{b}]^T \; \underline{\mu} &= 0 \\ \underline{\mu} &\geq \underline{0} \quad . \end{aligned} \tag{A13}$$

Rosen has assumed that the k constraints form a convex, bounded region R. Thus, it must be that $k \geq m+1$. Clearly, at most m of the k hyperplanes $\underline{\lambda}(\underline{x}^*) = \underline{0}$ are linearly independent. Let $\underline{n}_i$, $i=1, 2, \ldots, q$, be a set of linearly independent unit vectors which define a set of q linearly independent hyperplanes

$$\lambda_i(\underline{x}^*) = 0 \quad , \quad i=1,2,\ldots,q \quad ; \quad 1 \le q \le m \quad . \tag{A14}$$

If $\lambda_i(\underline{x}^*) > 0$ for all i, i=1,...,k , then $\underline{\mu}$ must equal $\underline{0}$. In this case $\underline{x}^*$ is an interior point of the region R and the necessary conditions reduce to

$$\nabla_{\underline{x}} f(\underline{x}^*) = \underline{0} \quad , \tag{A15}$$

the familiar necessary condition for an unconstrained problem.

Now assuming that $\lambda_i(\underline{x}^*) = 0$, i=1,... ,q, which can be the case for at most m of the k hyperplanes, the mxq matrix

$$N_q = [\underline{n}_1 \ \underline{n}_2 \ \cdots \ \underline{n}_q] \tag{A16}$$

can be formed from the vectors defining the above hyperplanes. Also, letting $\underline{\mu}_q$ be the vector of the μ_i which correspond to $\lambda_i(\underline{x}^*) = 0$, the Kuhn-Tucker necessary conditions can be written as

$$\nabla_{\underline{x}} f(\underline{x}^*) + N_q \underline{\mu}_q = \underline{0} \tag{A17}$$

$$\lambda_i(\underline{x}^*) = 0 \quad , \quad i=1,\ldots,q \tag{A18}$$

$$\underline{\mu}_q \ge \underline{0} \quad . \tag{A19}$$

Using the psuedo-inverse of N_q to find $\underline{\mu}_q$ from (A17) gives

$$\underline{\mu}_q = (N_q^T \ N_a)^{-1} \ N_q^T \quad [-\nabla_{\underline{x}} f(\underline{x}^*)] \quad . \tag{A20}$$

From (A19) this must be $\ge \underline{0}$, or

$$(N_q^T \ N_q)^{-1} \ N_q^T \ [\nabla_{\underline{x}} f(\underline{x}^*)] \le \underline{0} \quad . \tag{A21}$$

Equation (A17) can now be written as

$$\underset{\sim}{\nabla}_{\underline{x}} f(\underline{x}^*) + N_q(N_q^T N_q)^{-1} N_q^T [-\nabla_{\underline{x}} f(\underline{x}^*)] = \underline{0}$$

$$[I - N_q(N_q^T N_q)^{-1} N_q^T] \nabla_{\underline{x}} f(\underline{x}^*) = \underline{0} \quad . \tag{A22}$$

Using Rosen's definition of the projection matrix

$$P_q \equiv I - N_q(N_q^T N_q)^{-1} N_q^T \quad , \tag{A23}$$

equation (A22) becomes

$$P_q [\nabla_{\underline{x}} f(\underline{x}^*)] = \underline{0} \quad . \tag{A24}$$

Equations (A21) and (A24) are precisely the necessary conditions given by Rosen [15] in <u>Theorem 4</u> for $\underline{x}^*$ to be a constrained maximum of $f(\underline{x})$. If $f(\underline{x})$ is a concave function of $\underline{x}$ the Kuhn-Tucker conditions, and thus equations (A21) and (A24), are also sufficient for $\underline{x}^*$ to be a constrained global maximum of $f(\underline{x})$.

APPENDIX B

The Contraction Mapping Principle [13] can be applied to find sufficient conditions for a sequence of points obtained from the sequential minimization-maximization of a real-valued, concave-convex function with continuous and bounded second partial derivatives to converge to the saddle point.

A mapping A of the space R into itself is said to be a contraction if there exists a number $\alpha < 1$ such that

$$\rho\,(A\underline{x}, A\underline{y}) \leq \alpha\,\rho\,(\underline{x}, \underline{y})$$

for any two points $\underline{x}, \underline{y} \in R$, where R is an arbitrary metric space, and ρ denotes the choice of the distance in R. In this thesis ρ is defined as

$$\rho\,(\underline{x}, \underline{y}) = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{\frac{1}{2}} = \left\{ [\underline{x}-\underline{y}]^T [\underline{x}-\underline{y}]^T \right\}^{\frac{1}{2}} .$$

The functions examined in this thesis may all be represented by quadratic polynomials of the form

$$f(\bar{\underline{x}}, \bar{\underline{y}}) = \bar{\underline{x}}^T A \bar{\underline{x}} + \bar{\underline{y}}^T B \bar{\underline{y}} + \bar{\underline{x}}^T \underline{c} + \bar{\underline{x}}^T D \bar{\underline{y}} + \bar{\underline{y}}^T \underline{e} + g \quad . \qquad \text{(B1)}$$

This can be reduced by linear transformations to a simpler form by introducing new vectors $\underline{x}$ and $\underline{y}$ defined as

$$\begin{aligned} \underline{x} &= \bar{\underline{x}} + \underline{h} \\ \underline{y} &= \bar{\underline{y}} + \underline{k} \end{aligned} \qquad \text{(B2)}$$

where $\underline{h}$ and $\underline{k}$ are constant vectors to be determined. Substituting (B2) into (B1) gives

$$\begin{aligned} f(\underline{x}, \underline{y}) = {} & (\underline{x}+\underline{h})^T A(\underline{x}+\underline{h}) + (\underline{y}+\underline{k})^T B(\underline{y}+\underline{k}) + (\underline{x}+\underline{h})^T \underline{c} \\ & + (\underline{x}+\underline{h})^T D(\underline{y}+\underline{k}) + (\underline{y}+\underline{k})^T \underline{e} + g \end{aligned} \qquad \text{(B3)}$$

which can be written as

$$f(\underline{x},\underline{y}) = \underline{x}^T A\underline{x} + \underline{y}^T B\underline{y} + \underline{x}^T D\underline{y} + \underline{x}^T(2A\underline{h} + \underline{c} + D\underline{k})$$

$$+ \underline{y}^T(2B\underline{k} + \underline{e} + D\underline{h})$$

$$+ (\underline{h}^T A\underline{h} + \underline{k}^T B\underline{k} + \underline{h}^T\underline{c} + \underline{h}^T D\underline{k} + \underline{k}^T\underline{e} + g) \quad . \qquad \text{(B4)}$$

By setting the coefficients of $\underline{x}^T$ and $\underline{y}^T$ equal to zero, the values of $\underline{h}$ and $\underline{k}$ are determined from

$$2A\underline{h} + D\underline{k} + \underline{c} = \underline{0}$$

$$D\underline{h} + 2B\underline{k} + \underline{e} = \underline{0}$$

or,

$$\begin{bmatrix} 2A & D \\ D & 2B \end{bmatrix} \begin{bmatrix} \underline{h} \\ \underline{k} \end{bmatrix} = - \begin{bmatrix} \underline{c} \\ \underline{e} \end{bmatrix} \quad . \qquad \text{(B5)}$$

If the matrix $\begin{bmatrix} 2A & D \\ D & 2B \end{bmatrix}$ has an inverse, (B5) can be solved for $\underline{h}$ and $\underline{k}$. This matrix will have an inverse if A is positive definite and B is negative definite. Equation (B4) then has the form

$$f(\underline{x},\underline{y}) = \underline{x}^T A\underline{x} + \underline{y}^T B\underline{y} + \underline{x}^T D\underline{y} + K \qquad \text{(B6)}$$

where A is positive definite, B is negative definite, K is a constant, and f has a saddle point at the origin.

Allowing the variables $\underline{x}$ and $\underline{y}$ to be unconstrained, for a particular $\underline{y}$ the function f is minimized when $\nabla_{\underline{x}} f(\underline{x},\underline{y}) = \underline{0}$. The value of $\underline{x}$ at this point is

$$\nabla_{\underline{x}} f(\underline{x},\underline{y}) = 2A\underline{x} + D\underline{y} = \underline{0}$$

$$\underline{x} = -\tfrac{1}{2} A^{-1} D\underline{y} \qquad \text{(B7)}$$

Similarly, for any $\underline{x}$, the function f is maximized at

$$\nabla_{\underline{y}} f(\underline{x},\underline{y}) = 2B\underline{y} + D\underline{x} = \underline{0}$$

$$\underline{y} = -\tfrac{1}{2} B^{-1} D\underline{x} \quad . \tag{B8}$$

Let $\underline{x}_{i-1}, \underline{y}_i$ be an arbitrary point. Minimizing f with respect to $\underline{x}$, the next value of $\underline{x}$ ($\underline{x}_i$) will be

$$\underline{x}_i = -\tfrac{1}{2} A^{-1} D\underline{y}_i \quad . \tag{B9}$$

Maximizing $f(\underline{x}_i,\underline{y})$ with respect to $\underline{y}$ gives as the new value of $\underline{y}$

$$\begin{aligned} \underline{y}_{i+1} &= -\tfrac{1}{2} B^{-1} D\underline{x}_i \\ &= \tfrac{1}{4} B^{-1} DA^{-1} D\underline{y}_i \quad . \end{aligned} \tag{B10}$$

Minimizing $f(\underline{x},\underline{y}_{i+1})$ with respect to $\underline{x}$ obtains

$$\begin{aligned} \underline{x}_{i+1} &= -\tfrac{1}{2} A^{-1} D\underline{y}_{i+1} \\ &= \tfrac{1}{4} A^{-1}DB^{-1} D\underline{x}_i \quad . \end{aligned} \tag{B11}$$

Equations (B10) and (B11) can be written

$$\underline{z}_{i+1} = C\underline{z}_i \tag{B12}$$

where

$$\underline{z}_i = \left[\begin{array}{c} \underline{x}_i \\ \hline \underline{y}_i \end{array}\right]$$

and

$$C = \left[\begin{array}{c|c} \tfrac{1}{4} A^{-1} DB^{-1} D & 0 \\ \hline 0 & \tfrac{1}{4} B^{-1} DA^{-1} D \end{array}\right]$$

If $\underline{z}_{i+1}$ is closer to the origin (the saddle point) than $\underline{z}_i$, i.e., if $\|\underline{z}_{i+1}\| < \|\underline{z}_i\|$ where $\|\underline{z}_i\| = \left\{ \sum_{j=1}^{2n} (z_{ij})^2 \right\}^{\frac{1}{2}}$, the sequential min-max procedure will converge. This will be the case if (B12) is a contraction mapping. From page 45 [13], a sufficient condition for the mapping C to be a contraction is

$$\Sigma_i \; \Sigma_j \; a_{ij}^2 \leq \alpha < 1 \tag{B13}$$

where the a_{ij} are elements of C.

An alternative result may be derived from the definition of a contraction mapping. A mapping

$$\underline{z} = C\underline{z}$$

is a contraction if

$$\rho(C\underline{z}) \leq \alpha \, \rho(\underline{z}) \tag{B14}$$

where $\alpha < 1$, that is,

$$\underline{z}^T C^T C \underline{z} \leq \alpha \; \underline{z}^T \underline{z} \tag{B15}$$

Thus, if

$$\underline{z}^T C^T C \underline{z} < \underline{z}^T \underline{z} \tag{B16}$$

then the mapping C is a contraction. Conversely if

$$\underline{z}^T C^T C \underline{z} > \underline{z}^T \underline{z} \tag{B17}$$

then C is not a contraction mapping.

It is well-known that on the sphere $\|\underline{z}\| = 1$, the maximum value of $\underline{z}^T C^T C \underline{z}$ is

$$\max_{\underline{z}} \underline{z}^T C^T C \underline{z} = \lambda_1 \tag{B18}$$

where λ_1 is the largest eigenvalue of $C^T C$, and that the minimum value is

$$\min_{\underline{z}} \underline{z}^T C^T C \underline{z} = \lambda_N \tag{B19}$$

where λ_N is the smallest eigenvalue of C^TC. The matrix C^TC is real, symmetric and positive definite, hence its eigenvalues are all real and positive.

From (B18) and (B19)

$$\lambda_N \; \|\underline{z}\|^2 \leq \underline{z}^T C^T C \underline{z} \leq \lambda_1 \; \|\underline{z}\|^2 \quad . \tag{B20}$$

Looking at the right-hand inequality first

$$\underline{z}^T C^T C \underline{z} \leq \lambda_1 \; \|\underline{z}\|^2 \quad , \tag{B21}$$

if $\lambda_1 < 1$ then it follows that

$$\underline{z}^T C^T C \underline{z} < \underline{z}^T \underline{z} = \|\underline{z}\|^2 \tag{B22}$$

and the mapping C is a contraction.

From the left-hand inequality

$$\lambda_N \|\underline{z}\| \leq \underline{z}^T C^T C \underline{z} \tag{B23}$$

and if $\lambda_N > 1$,

$$\underline{z}^T C^T C \underline{z} > \underline{z}^T \underline{z} \tag{B24}$$

and C is not a contraction mapping.

Thus, $\lambda_1 < 1$ is a sufficient condition for C to be a contraction mapping and $\lambda_N > 1$ is a sufficient condition for C <u>not</u> to be a contraction mapping.

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13. ABSTRACT

Nonlinear programming methods are used to solve saddle-point problems subject to inequality constraints on the variables; in particular, the type of saddle-point problem arising in pursuit-evasion differential games is considered. The methods investigated fall into two groups: solution of the nonlinear simultaneous equations obtained from the Kuhn-Tucker conditions, and solution of a sequence of constrained optimization problems by the gradient projection algorithm. These methods are applicable to any real-valued function $f(\underline{x},\underline{y})$ which is convex in $\underline{x}$, concave in $\underline{y}$, and has continuous and bounded second partial derivatives. Several examples are given which illustrate the characteristics of the numerical procedures.

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